The Golden mean, scale free extension of Real number system, fuzzy sets and $1/f$ spectrum in Physics and Biology

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Abstract

We show that the generic $1/f$ spectrum problem acquires a natural explanation in a class of scale free solutions to the ordinary differential equations. We prove the existence and uniqueness of this class of solutions and show how this leads to a nonstandard, fuzzy extension of the ordinary framework of calculus, and hence, that of the classical dynamics and quantum mechanics. The exceptional role of the golden mean irrational number is also explained.

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1 Introduction

The ubiquitous presence of $1/f$ spectrum in a wide range of Natural and biological processes[1, 2] is considered to be an interesting problem in theoretical physics. Any new insight into the problem is expected to enhance our understanding of the origin of complex structures in Nature. In this paper, we present a body of new mathematical results which is likely to initiate a new approach in understanding the origin of complexity and the generic $1/f$ spectrum problem in Nature.

In a recent paper [3], we have initiated an investigation which explores the role of time inversion in the context of a linear differential dynamical system. By 'time inversion' we mean the following. We assume that time has a stochastic element in its small scale behaviour, viz.; time can 'fluctuate' following the inversion rule $t_- = 1/t_+$, where $t_- = 1 - \eta$, $t_+ = 1 + \eta$, $0 < \eta << 1$, and η may be a stochastic variable, in the neighbourhood of $t = 1$, say. The ordinary linearly flowing (reversible) time sense of order $t \sim$O(1), then , arises only in the mean, when the small scale stochastic fluctuations are properly *coarse grained*. Our investigations under this assumption then lead to the uncovering of an exact class of scale free solutions to the linear equation

$$\frac{\mathrm{d}\ln T}{\mathrm{d}t} = 1 \tag{1}$$

We show that thanks to this exact class of solutions, the real number system R is identified with its nonstandard extension [4] $\mathbf{R}$, viz.; $\mathbf{R} = R$. Consequently, every real number is identified with an equivalence class of scale free fluctuations, undergoing stochastic, irreversible evolutions following the cascades of the infinite continued fraction of the golden mean $\nu = (\sqrt{5} - 1)/2$. We show that this scale free fluctuation of the stochastic fractal time endows a linear system with the generic low frequency $1/f$ spectrum. Further, the generic probability distribution of these scale free fluctuations turns out to be a Gamma distribution.

The relevance of the present work may be judged in the light of the recent works [5, 6] uncovering new relationships between time and the number theory. In ref[7], Selvam, on the other hand, developed a cell dynamical model for the computational error growth, analogous to the formation of turbulent eddies. It is shown that the ratio of higher to lower precision error domains scales, in a long computer run, as the golden mean ν, revealing a universal self similar pattern in the round- off error structure. Further, the theory of Cantorian spacetime of El Naschie [8-12] seems to anticipate some of the features of the scale free fractal time.

In the present paper, we reverse our previous approach and base our analysis directly on the exact scale free solutions of eq(1). Our main results in the form of certain theorems are proved in the next section. We show that the real number system is a fuzzy, nonstandard set, infinitesimally small elements of which experience a stochastic, irreversible evolution. In Sec.3, we show how these intrinsic fluctuations lead to the generic $1/f$ spectrum for a general differential dynamical system. We close in Sec.4 with some concluding remarks.

2 Main theorems and their proofs

Our main results are stated in the following three theorems.

Theorem 1 *There exists a unique class of one parameter family of scale free solutions*

$$\ln T(t) = t + k\frac{T(\ln t)}{t},\ t \neq 0 \tag{2}$$

k being an arbitrary parameter, to eq(1). These solutions are fundamentally non-local.

Consequently,

Theorem 2 *This non-locality directs a real variable t to change, in the neighbourhood of $t = 1$, by a 'local inversion', $t_- = 1/t_+$, where $t_- = 1 - \eta$, $t_+ = 1 + \eta$, $0 < \eta << 1$, in contrast to the conventional mode of linear increments. Consequently, the solutions are only continuously first order differentiable, at the points where t changes by local inversions.*

Theorem 3 *The family of solutions is also shown to provide a 'dynamical' representation of the real number system, each member of which is identified with an equivalence class of scale free fluctuations, undergoing a stochastic, irreversible evolution, following the cascades of of infinite continued fraction of the golden mean. Consequently, the real number set R is a fuzzy, nonstandard set.*

The existence part of Theorem 1 is almost trivial. To verify that eq(2) is indeed a solution, we consider instead a more general ansatz $\ln T(t) = t + k\frac{\tau(t)}{t}$, and assume that both T and τ are continuously first order differentiable in t. A direct differentiation now yields $\frac{\mathrm{d}\ln T}{\mathrm{d}t} - 1 = \frac{k}{t^2}(-\tau + t\frac{\mathrm{d}\tau}{\mathrm{d}t})$, which is true for all $t\,(\neq 0)$ (and k). One thus obtains eq(1), along with the fact that $\tau(t) \equiv T(\ln t)$, since

$$t\frac{\mathrm{d}\ln\tau}{\mathrm{d}t} = 1 \tag{3}$$

We note that $T(t)$ is indeed a new solution, since it is undefined at $t = 0$. Clearly, the standard solution of eq(1) $\ln T_s = t$, $T_s(0) = 1$ follows from the above ansatz provided we choose $k = 0$. However, in the present framework, there is no reason for this 'a priori' restriction. The form of τ means that when T is a function, for instance, of $t \gtrapprox 1$, then $\tau(t)$ must be a self similar replica of T on the smaller logarithmic scale $\ln t$, so that $\tau(t \approx 1) = T(t \approx 0)$. The total solution $T(t)$ is thus obtained as a co-operative effect of two different scales that arise naturally in eq(1), viz., when the variable t is allowed to change from $t \gtrapprox 0$ to $t \approx 1$, and then is mapped back to $t \gtrapprox 0$ on the logarithmic scale: $dt \to d\ln t$ near $t \approx 1$ (c.f., remark 1). Indeed, eq(3) coincides with eq(1) in the limit $t \to 1$. If one wishes, instead, to write $\tau = ct$, $t > 0$ as usual, then eqs(1) and (3) match over the *ordinary scale* $t \sim$O(1), $\ln t \sim$O(1), viz., $1 \lessapprox t \sim$O(e), provided $k \approx 0$, $c \approx 1$, when the initial condition is defined at $t = 1 : \ln T(1) = 1$ (imposing initial condition at $t = 0$ is not allowed since $t \neq 0$). The standard solution T_s is thus realized as a particular solution, when this more general class of solutions is restricted to the ordinary scale $t \sim 1$. In anticipation of the following, we call the nontrivial part (T_f) in T as the 'fluctuation' over the standard mean solution T_s. Further, the solution is, in general, non-local, because $T(t)$ defines a non-local ('distant') connection between two logarithmically separated points on the t- axis, for instance, $t = 1$ and $t = 0$ (equivalently,$t \to 0^+$ and $t \to \infty$). The non-locality drops out (approximately) only over the ordinary scale.

Remark 1 We note that the standard general solution of eq(1), but restricted to the punctured real line $R - \{0\}$, viz.; $\ln T = t + c$, c a constant number, and $t \neq 0$, could be interpreted to belong to the generalised class of solutions of eq(1), for an unrestricted t, and vice versa, since $\ln T(t) = t + k\frac{\tau(t)}{t}$, $c = kc_0$ and $\tau = c_0 t$ is the standard solution of eq(3). Because of the translation symmetry of eq(1), the initial value problem in either of the cases is defined only approximately at the initial point $t \approx 0$. The constant c is thus determined only approximately, which then acts as a seed for the residual self similar evolution for the supposedly constant fluctuating component T_f.

Remark 2 An alternative way of seeing the possibility of a nontrivial fluctuation for a solution of eq(1) is to use the Born-Oppenheimer- like factorisation [13] $T = T_s T_f$ so that eq(1) reduces to $t^{-1}\frac{\mathrm{d}T_f}{\mathrm{d}\ln t} = 0$. Clearly, for any moderately large value of $t(> 0)$ T_f is a constant, but for an arbitrarily large $t:\ t^{-1} = \epsilon \approx 0$, $\frac{\mathrm{d}T_f}{\mathrm{d}\ln t}$ can indeed be O(1).

To study the salient features of the solution, we note that the standard solution $\ln T_s = t$ defines a 1-1 (identity) mapping $R \to R$. Let $\phi(t) = \tau(t)/t$, $\tau(t)$ being a nontrivial solution of eq(3). Then the new solution eq(2) gives an extension of T_s to a class of scale free solutions, since τ, being a solution of eq(3), is scale free. We note that the fluctuation $\phi(t)$ is also scale free, since ϕ satisfies the scale free equation

$$t\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \tag{4}$$

and hence represents a 'slowly varying constant'. Both τ and ϕ satisfy the scaling law

$$f(kt) = kf(t) \tag{5}$$

where $f(t)$ is a scale free function. The scaling law for τ follows from eq(3). Consequently, $k\tau(t)/t = \tau_1(t_1)/t_1$, $t_1 = kt, \tau_1 = k\tau$, and hence ϕ also satisfies eq(5). It also follows that ϕ is *a 'universal' function, defined in the vicinity of* $t = 1$. Because of the translation symmetry $t \to t - t_0$ of eq(1), the fluctuation of $T(t)$ is actually defined pointwise $\ln T(t) = t + k\phi(t')$, $t' = t - t_0$. By the above universality, however, $\phi(t') = \phi(t)$, and hence the 'nontrivial' neighbourhood, denoted 't', of every real number t is identical to '1': 't'-$\{t\}$='1'-$\{1\}$. Here, $\{t\}$ is the singleton set of the real number t. Consequently, $\phi(t)$ represents the universal scaling behaviour of the neighbourhood of every real number.

Let c denote a constant solution of eq(4), in the ordinary sense. Then a scale free extension is given by $\phi_c(t) = c(1 + k\phi_c(\ln t))$. Rescaling $\phi \to \phi_c/c$ and $k \to kc$, and using the scaling law eq(5), we get $\phi \in$'1' in the form $\phi(t_1) = 1 + k\phi(\ln t_1)$. Here, t_1 is another 'universal local parameter', defined in '1'. In-fact, by the inversion symmetry of eq(3), viz.; $\frac{\mathrm{d}\ln\tau}{\mathrm{d}\ln t} = 1 \Leftrightarrow \frac{\mathrm{d}\ln t}{\mathrm{d}\ln\tau} = 1$, t can be treated as a function of τ, and thus $t_1(t) = t(\tau(t)) \in$'1'. The logarithm in the k-dependent term, as stated above, signifies non-locality, if the lhs is defined at $t_1 = 1$, then the rhs is defined at $\ln t_1 = \epsilon \approx 0$ and hence at $t_1 \approx 1 + \epsilon$, by universality.

Now, in relation to the scale defined by eq(1), any real number t is written in the scale free representation as $t_f(k) = t\phi(t_1)$. Clearly, t_f satisfies eq(3), and defines a one parameter family of extensions of the identity mapping, $1_f : R \to R$. However, $1_f = \phi(t_1)$, and hence the scaling parameter k must be sufficiently small $0 < |k| << 1$. *The extension is also unique*, for each choice of a fixed, but arbitrarily small, k. For, let $\bar{\phi}(t)$ be another solution of eq(3). Then, $|\phi(t) - \bar{\phi}(t)| = |k||\phi(\ln t) - \bar{\phi}(\ln t)| = \epsilon|\phi(\bar{t}) - \bar{\phi}(\bar{t})|$, $\bar{t} = (k\ln t)/\epsilon$, by the scaling law eq(5), for any small $\epsilon > 0$, so that $\bar{\phi} = \phi$, again by universality. This completes the proof of Theorem 1.

Remark 3 The nontrivial 'slowly varying constant' $\phi(t)$ in the generalised solution eq(2) tells the impossibility of erasing a *residual* t dependence not only from the fluctuating component of the solutions of eq(1), but also for any real number. In the following, we show that this residual t-dependence has an intrinsic time-like feature. In ref[3], we have shown how *the golden mean partition of unity* :$\nu^2 + \nu = 1$, $\nu > 0$ could be utilised to magnify a practically insignificant time dependence to an observable effect over a sufficiently long time scale. In fact, the residual fluctuation in log scale, which remains *unnoticed* in the usual treatment of Calculus, gets manifested under a nontrivial SL(2,R) realization of a linear translation

$$t = 1 + \eta \approx \frac{1 + \nu^2 \eta}{1 - \nu \eta}, \tag{6}$$

for a sufficiently small $0 < \eta << 1$. A fluctuation which is negligible in the scale of $t \approx 1$ can grow to O(1) when $\ln t \approx \eta \sim$O(1). The presence of a residual time dependence could also be the reason for the origin of $1/f$ noise in the prime number distribution [5].

Remark 4 In the usual treatment, the scaling law eq(5) means $f(t) = tf(1)$. Clearly, this is derived under the assumption that the equality $kt = 1$ is exactly realized. In the context of the generalised solution eq(2), the validity of an exact equality (as in the standard classical analysis) is violated. A key reason of this violation is the non-locality of the solution, so that the exact value of $T(t)$ at a particular t is non-computable since this requires a string of recursions spanning over a number of distinct logarithmic scales (for instance, $\ln t$, $\ln\ln t, \ldots$ for a sufficiently large t). Consequently, the set defined by $T(t)$ is a fuzzy set and hence the real set R itself is fuzzy. This proves partially Theorem 4. (For a more precise sense of fuzziness, see below.)

Now, to prove Theorem 2, we note that $1_f \in$'1', by definition, so that $\phi(t_1) = 1_f = t_1$, by the uniqueness of the extension. We thus get

$$t_1(t) = 1 + kt_1(\ln t) \tag{7}$$

This also justifies the term 'slowly varying constant', since for an arbitrarily small $|k|$, $t_1 \approx 1$, is almost constant over the scale of $t \sim 1$, but undergoes (infinitely) slow variation, (c.f., Remark 3) which gets manifest in the smaller scale $\ln t \sim 0$: $dt_1/dt \approx 0$, but $dt_1/d\ln t \sim$O(1).

Now, to see how *a nontrivial definition of an inversion follows from eq(2) directly, leading to the 2nd derivative discontinuity,* let $t_- = 1 - \eta$, $t_+ = 1 + \eta$, for $0 < \eta << 1$. Then we have

$$t_1(t_-) = 1 + kt_1(\ln t_-) = 1 - kt_1(\ln t_+) \tag{8}$$

by the scaling law eq(5), and

$$t_1(t_+) = 1 + kt_1(\ln t_+) \tag{9}$$

Hence, even as t is assumed to change from t_- to t_+ by translation (say), $t_1(t_-)$ gets linked *instantaneously* to $t_1(t_+)$, as in eq(8), because of the non-local connection. This instantaneous 'distant connection' between two distinct points in the t scale tells that the change in the zeroth order scale t, near $t = 1$, to avoid any paradoxical situation (viz., reaching the destination before being started), must have been accomplished instantaneously, instead, by an application of a local inversion for a small enough η. This zeroth order inversion then induces analogous mode of inversions even for the small scale variable t_1, since $t_1(t_+) = 1/t_1(t_-)$ for $(0 <)k << 1$. It thus follows that *a local inversion must replace the ordinary translation in accomplishing changes in the vicinity of a point on the real axis,* for the sake of consistency. It is now easy to verify the 2nd derivative discontinuity of the scale free solution of eq(3), by differentiating eq(8) and eq(9), and then taking the limit $\eta \to 0^+$. Clearly, the first derivatives matches continuously at $t = 1$, but there is a mismatch in sign in the second derivatives. This proves Theorem 2.

We note that this discontinuity is not in contradiction with the Picard's theorem[14]. The framework of *the Picard's theorem (and, as a matter of fact, that of the (real) analysis) gets*

extended under the scale free properties of real numbers and local inversions. We give a thorough independent analysis of this theorem elsewhere.

We now proceed to prove Theorem 3. We do this in several steps. First, we show why *the above discontinuity ought to be be removed to a very small (large) value of t.* We begin by showing, in an alternative way, how the extension of the real system R defined by the generalized mapping 1_f *forbids* an *exact* evaluation of a real number (c.f.Remark 3). We note from eq(3) that

$$\tau(t_-) = \tau(t_+^{-1}) = 1/\tau(t_+) \tag{10}$$

which follows by replacing t by t^{-1} in the equation and identifying the '-' sign in the two alternative ways. Further,

$$\tau(t_+)/t_+ = t_-\tau(t_-^{-1}) = t_-/\tau(t_-) \tag{11}$$

and hence

$$\phi(t^{-1}) = 1/\phi(t), \Rightarrow \phi(t_-) = 1/\phi(t_+) \tag{12}$$

Clearly, these relations are valid even in $\ln t$. Consequently, we have $t_1(t_-) = 1 + kt_1(\ln t_+^{-1}) = 1 + k/t_1(\ln t_+)$, where $t_1(\ln t_+) = 1 + kt_1(\ln(\ln t_+)^{-1})$. We note that near $t = 1$, $0 < \eta << 1$, so that $t_1(\eta) = 1 + kt_1(\ln(\eta)^{-1})$ and hence in the limit $\eta \to 0^+$, we get $t_1(0) = 1 + t_1(\infty)$, by the scaling law eq(5), since $k\infty = \infty$, for *any* finite k. Further, by definition, 0 and ∞ are related by an inversion, so that $t_1(0) = 1+1/t_1(0)$ and $1/t_1(\infty) = 1+t_1(\infty)$. We thus have $t_1(0) = 1+\nu$ and $t_1(\infty) = \nu$, where $\nu = (\sqrt{5} - 1)/2$, is the golden mean irrational number. Apparently it means that the *exact* value of the golden mean is realized as the limit of the universal scale free solution of eq(3), as t approaches either ∞ or 0, when the inversion $t_1(\infty)t_1(0) = 1$ is exactly satisfied. In reality, however, *the realization of the exact value of ν is impossible, in principle.* (We note that the local inversion is valid upto $O(\eta^2)$.)

Now, the above assertion follows if one assumes the existence of a nontrivial *infinitesimally small number*, in the sense of the nonstandard analysis[4]. In fact, *we show that such a number exists.* Let the solution of eq(3) be written as $t_f = t + k\phi(t) = t(1 + k\phi(t)/t)$.[1] It follows from eqs(7,12) that

$$\phi(t^{-1}) = 1 + \tilde{k}\phi(t),\ \tilde{k} = k/t \tag{13}$$

Now, for each $0 < t < 1$, there exists a k so that $0 < \tilde{k} < t$. So, in the sense of a limit in the ordinary calculus, $\tilde{k} \to 0$, as $t \to 0$, leading to a contradictory result that $(\phi(\infty) =)\nu = 1$. This contradiction, however, is removed in the context of a 'physical' limit, which allows a real variable t (say) to approach 0 (say, but *never coinciding* with 0), by *exploring smaller and smaller scales.* It turns out that this exploration of scales is a never ending process, and so, in principle, *delays the realization of the final limit to an infinitely distant epoch.* We note that limit (and for that matter, any continuous process of change) is basically a *dynamical* concept, and must have an intrinsic correlation with a sense of time. The continuous (monotonic) change of t to smaller and smaller values (scales) can indeed be correlated to an increasing sense of time through an iterative process. (In ordinary treatment, t traverses the interval $[0, 1] \downarrow$ in unit time. In the context of the scale free solution of eq(1), the relevant interval is (0,1] and as argued below, the approach $t \to 0^+$ should, in principle, be a never ending process (c.f., the concluding remarks).) We now show that (A) *the small scale variable $t_1 = (\phi(t))$ is intrinsically stochastic with an irreversible arrow of 'time'.*

[1]Note that the scale free extension of R is two-fold, viz.; $t_{f+} = t\phi(t)$, $t_{f-} = t\phi(t^{-1})$, for each $t \in R$ so that $t_{f+}t_{f-} = t^2$, for each $k > 0$.

Indeed, we note that (1) $\tilde{k}$ can be a 'slowly varying constant', for an *infinitesimally small*, slowly varying $k(t)$ satisfying eq(2). Consequently, as $t \to k$ (t varies faster in comparison to k, and must cross k in a 'finite' elapse of time), the applicability of the ordinary sandwich theorem is violated, because of the reversal of the above inequality: $(k \sim)t < \tilde{k}(\sim 1)$, allowing a local inversion to materialise, in the neighbourhood of the slowly varying k. Further, (2) the scale free representation

$$\nu = \nu_f/\phi(t^{-1}) \tag{14}$$

tells that *the golden mean number ν must be identified with a uncountable set of fluctuating, approximate values (evaluations) of the said number.* Consequently, as t approaches smaller and smaller scales, an approximate t-dependent value of ν, denoted $\nu_{[t^{-1}]}$, $[t^{-1}]$ being the greatest integer value function, and given by eq(14), now approaches slowly to more and more accurate evaluations of ν through the sequence of convergents $\{\nu_n\}$ of the golden mean continued fraction. Indeed, we have $\nu_n = \nu_f/(1+(k(t)/x_n t))$, $x_n = \nu_f/\nu_n$, $0 < k << t^2$, for a sufficiently large $n = [t^{-1}]$, but $\nu_n = \nu_f/(1+1/x_n)$, at $t \sim k$, so that $\nu_{n+1} = \nu_f/(1+1/(1+(k(t_1)/x_{n+s}t_1)))$, where $t_1 = kt$, t being a decreasing O(1) variable from $t = 1$, and so on. We note that the self-similarity of intervals (k^2, k) and $(k, 1)$ tells that there exists a $0 < k_1(t) = k(t_1) << t_1^2$ so that $x_n = 1 + (k(t_1)/x_{n+s}t_1)$, $n + s = [t_1^{-1}] \approx [k^{-1}]$, $\nu_{[t_1^{-1}]} = \nu_{n+s}$. It thus follows that, for $\nu_f = \nu_{n-1}$, eq(14) represents an intrinsic evolution of the convergents ν_n from a lower precision to higher and higher precision values, as t tends to 0 slower and slower, since $dt_1/dt << 1$. Further, any approximate value ν_f of ν can be rescaled to an convergent: $c\nu_f = \nu_{n-1}$ so that $\nu_n = \nu_{n-1}/\phi(ct^{-1})$.

Let us note here that k/t remains arbitrarily small, but non-zero, even in the limit $n \to \infty$. Consequently, even as all the ordinary positive real numbers, represented by the scale t, are exhausted in the limit $t \to 0^+$, k/t remains dynamically active, although $k = 0$ in the limit. We thus *define an infinitesimally small* $k(> 0)$ by the condition that $[t/k] = s$ be arbitrarily large as $t \to 0^+$. Clearly, an infinitesimal k *exists*, by the above construction. Obviously, K is *infinitely large* if $1/K$ is infinitesimal. Further, infinitely large and infinitesimally small numbers are (relative) scale dependent, for instance, k is infinitesimal relative to the scale $t \sim$O(1) $\Leftrightarrow$ k is infinitely large relative to $t \sim$O(k^2).

Now, returning to the golden mean convergents, we note that as t explore the nth order smaller scale, the unfolding of the nth level cascade of the continued fraction is activated. Thus ν_n tends to the exact value ν, as t^{-1} approaches ∞ exploring *infinitely* larger and larger scales. But, *this unfolding of higher order cascades must be a never ending process*, since, by the self-similarity of real axis over scales, there can not be a largest infinitely large real number. Thus, *any* approximate value of ν must experience an intrinsic evolution towards the exact value, but would never be able to attain the final accuracy. We remark that although there is no exact value of ν in the present formalism, a progressive evolution towards better and better accuracy values still makes sense. As stated before, any iterative process generates a *directed sense of time*. However, the exact injection moments of local inversions, leading the evolution to multiple scales, is uncertain, because of (a) the arbitrariness in the sign of the infinitesimal scaling constant k and (b) the absence of an exact value. Hence, the intrinsic time sense is *stochastic* as well. This proves our assertion (A).[2] In the following, we, however, disregard the randomness from (b) (which will be considered separately).

[2] The directed evolution is global in the sense that every real number evolves from a lower to higher precision values following the sequence of the golden mean convergents. This global irreversibility should also be addressable in the sense of the informational complexity (entropy) (or in terms of the Hausdorff dimension in $\mathcal{E}^{(\infty)}$ spaces (see e.g.,M. S. El Naschie, Chaos, Solitons & Fractals, 2002, 14(7), 1121)). We defer this approach

Further, every real number r can be written as $r = r_0\nu$, for a suitable r_0, so that the above analysis applies to every $r \in R$. Consequently, the scale free eq(2) can indeed endow every real number r with an intrinsically time dependent, scale free representation $r_f(t)$, which is essentially a (continuous) distribution of approximate values (fluctuations) undergoing a slow, stochastically scale free, cascaded evolution down the infinite staircase of the golden mean continued fraction. We note also that a real number r belongs to the equivalence class of any other number, for instance, 1 say, defined by the scale dependent (interval-valued fuzzy) membership function[15]: $r_f = r\phi(t_1^{\pm 1}, k)$. Clearly, an approximate evaluation r_f of r belongs to the fuzzy number 'r_f' with a membership value $\phi(t_1^{\pm 1}, k) \lesssim 1$, for $t_1 \sim 1$ but belongs to the fuzzy '1' with a membership $0 < \phi(rt_1, k) < 1$, if $0 < r < 1$, and $0 < 1/\phi(rt_1, k) < 1$, if $r > 1$, for $t_1 \sim 1/r$. A detailed study of the fuzzy aspects will be considered separately. This completes the proof of Theorem 3. We note that the nonstandard extension of the real number system constitutes a valid model of analysis[4]. *The self-consistent derivation of a nontrivial solution on the basis of the scale free eq(3) therefore reveals hitherto unexplored new features of the real number system, thereby elevating the possible solutions to eq(1) to the class of finitely differentiable solutions.* We note that infinite differentiability on the scale t reappears in the present context, when the realisation of inversion is removed to an infinitely distant moment $t = \omega$. However, for any t there exists k so that $kt \approx 1$. Thus, this postponement of inversion, and consequent scale changes, could not be maintained for an indefinitely long period of time, involving many infinitely large scales. This proves the *inevitability* of the scale free solutions in the context of eq(1).

3 Power spectrum

We note that the randomness in sign raises t_f to a random variable $\mathbf{t_f}$. Since both the signs are equally likely, the expectation value of $\mathbf{t_f}$ is written as $< \mathbf{t_f} >= t(1 + \tilde{k}(t_1)\phi(t_1))$, $\tilde{k}(t_1) = (k(t_1) + k(1/t_1))/2$, which has the same form of a nonrandom t_f with $k(> 0)$ replaced by $\tilde{k}$. We note that the small (late) t asymptotic form of the fluctuation in the scale free solution eq(2), and that of $< \mathbf{t_f} >$ is given by $\bar{t}_f \sim t^{\mu}$, $\mu = k\frac{\phi}{\ln t}$, according as $t \to 0$ or ∞. Here, $\bar{t}_f = t_f/t$ denote the (total) scale free fluctuation of eq(2), over the scale t, when the standard mean component is removed. The above universal fluctuation pattern is obtained simply by rewriting the ansatz for the scale free solution for eq(2). The large t asymptotic form of the correlation function of the fluctuation spectrum is now written as $C(t) =< \bar{\mathbf{t}}_\mathbf{f}\bar{\mathbf{t}}_\mathbf{f}(\mathbf{0}) >\approx t_f^2 \sim t^{2\mu}$, since $\bar{t}_f(t) = \bar{t}_f(1)$, $\forall t$, by the universality of the scale free solution.

We remark that in the framework of the classical measurement hypothesis, any real physical variable is exactly measurable. The above analysis shows that this hypothesis is violated because of the irreducible scale free fluctuations. Further, in a realistic physical application, any physical variable T (say) is measurable with a finite degree of accuracy, say $r << 1$, only. The infinitesimal scale free fluctuations in the exact equation (3) thus correspond to the uncertainties in the measured values of this physical variable. In that case, the actual infinitesimally small scaling parameter k, behaves as a 'relative infinitesimal', viz., when k is any real number smaller than the accuracy limit, $k < r$. The scale free dynamical representation of real numbers now tells that any relatively negligible fluctuation would grow in time, and would have nontrivial dynamical effects, as reflected in the non-zero late time asymptotic exponent μ. The power spectrum of this power law asymptotic correlation function has the generic form $1/f^{1+2\mu}$. We

for a separate investigation.

note that a small, but nonzero, μ is sufficient to generate the generic $1/f$ spectrum.

Finally, a general (differential) dynamical system, represented by an ODE of the form $f(t, x, \dot{x}, \ddot{x}, \ldots) = 0$, where t is the ordinary time t, is, in view of eq(3), an equation, essentially, in t_f. The evolution of the system thus inherits the scale free properties of time itself, apart from any special features depending on the explicit nonlinearities in the problem, the signature of which should be revealed in the form $\mu = \mu_f + \mu_d$, where $\mu_f = k\phi$ is the universal component from the scale free time and μ_d is a model dependent term. Thus the universal presence of $1/f$ spectrum in the Natural processes over a long time scale is adequately explained in the framework of the scale free solutions of the linear ODE eq(1).

4 Closing remarks

We conclude with the following remarks.

1. The conventional linear treatment of the limit $t \to 0$(say) overlooks the presence of nonlinear slowly varying scales of the form $0 < k(t)t << t << 1$, because of the tacit assumption that t in $0 < t << 1$ approaches 0 with the uniform speed 1. Consequently, 0 in the ordinary real number system corresponds to $0_f = 0^+ \cup 0^-$, $0^\pm = \{\pm k(t)t, 0 << t << 1\}$, since the infinitesimal scales are indeed *unobservable* in the framework of the linear Calculus.

2. Let $I(t) = (-t, t)$, $t \to 0^+$. Then the cardinality of $I(t)$ is c, the cardinality of continuum. However, cardinality of $I(0) = 0$. This abrupt discontinuous transition from c to 0 is not explained in the linear Calculus. The scale free extension 0_f of the ordinary 0, on the other hand, is a Cantor -like set with scaling exponent μ, which, for $t \to 0^+$ is identified as the uncertainty exponent [16] for the fattened real number set R_f. However, the scale free extension, being defined by an exact solution of eq(1) means that $R_f \equiv R$, as remarked already. Thus every point of the real line is structured and accommodates both the point-like and continuum properties analogous to a Cantor set. The ordinary point -like (non-fuzzy) structure arises under an approximation, viz., when the continuum of nonlinear infinitesimal scales are neglected. The small, but non-zero uncertainty exponent gives a measure of the limit of accurate evaluation (measurement) of a physical quantity; since the uncertainty exponent μ tells that points separated by a distance less than μ would be indistinguishable.

3. In a dynamical system, the ordinary time t is obtained as the (zeroth order) "mean field" realization of the scale free time t_f, as the expectation value of the infinitesimal scale free fluctuation $k\phi(t)$ is negligible. The ordinary evolution of the system would, therefore, continue till $t \sim 1/k$, k being the supremum of infinitesimal scales. During the transition period (i.e., when $t \to (1/k)^-$), the system would experience random fluctuations as the scale free component (fluctuation) grows to O(1): $\phi(kt) = \phi(t_{1-}) \sim 1$. The system, however, returns to the ordinary regular pattern of evolution once t crosses $1/k$ from left to right as $\phi(t_1^{-1}) = 1 + k\phi(t_1)$ and t_1 tends slowly (compared to t) to $(1/k)^-$. The scale free evolution of a system would thus resemble, generically, to an intermittent process as t tends to ∞ exploring longer and longer scales. Any dynamical system, when evolved over an 'infinitely' long period of time, would inherit naturally this universal pattern of fluctuations of the scale free time. Clearly, this universal fluctuation is obtained as a consequence of a cooperative effect of random infinitesimal scales, which contribute gradually as the evolution is continued over longer and longer periods of time. Consequently, even a simple system such as eq(1) tends to behave as an extended system (random scales behaving as independent degrees of freedom), when the system is allowed to evolve over longer and longer time scales. The scale free extension of

Calculus, and hence of classical (and quantal) dynamics, therefore, appears to provide a natural framework for a deeper analytical treatment of a large body of natural processes which exhibit the phenomena of self-organised criticality [2]. We note further that the present formalism is also expected to offer a new approach in dealing with quantum-classical transitions and other related issues [17], since the scale free extension can already be considered to be a non-classical extension of the classical dynamics.

4. It turns out that the Cantorian spacetime theory of El Naschie [8-12] has already anticipated some of the salient features of this extended *dynamical* Calculus. We mention, in particular, of (i) a duality (inversion $t \to 1/t$) transformation used to obtain the Hausdorff dimension of the $\mathcal{E}^{(\infty)}$ space as $\phi^{-3} = 4 + \phi^3 = 4.2360679...$ [8] from the Fisher's scaling relation involving low energy critical exponents (see also [9]) and (ii) the non-locality in the sense that " a 'point' particle may be at two different spatial 'locations' at the same 'time' " [10]. This non-locality is a consequence of the indistinguishability between intersection and union in the underlying $\mathcal{E}^{(\infty)}$ space. It is interesting that gamma distributions appear naturally in the probabilistic theory of such a spacetime [11]. Further, the scaling exponents α of $1/f^\alpha$ noise fluctuations observed in semiconductor materials and quasi-crystals are related to the Hausdorff dimensions of backbone Cantor sets in $\mathcal{E}^n$ with mean topological dimensions n=4 and 5 respectively [12].

5. The singular role of the golden mean in the present dynamical formalism of Calculus suggests that a signature of the golden mean must be imprinted deep at the heart of every dynamical process in Nature.

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